Universal scaling law of the power spectrum in the on-off intermittency

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The power spectrum of the solvable on-off intermittency model previously introduced by the authors is analytically derived. A scaling law holds in the neighborhood of the critical point. Its universality is numerically confirmed for coupled maps and a stochastic model. [S1063-651X(98)14111-9]

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Fujisaka and Yamada first found an intermittent behavior in coupled chaotic oscillators, which occurs when the synchronized state is broken [1]. In their seminal papers this intermittency was called *type B intermittency* or *intermittency caused by chaotic modulation*. However, the name *on-off intermittency* is mostly used in the current literature. This phenomenon has been observed in various nonlinear systems [1–10]. When the synchronized state is stable, the chaotic attractor is located in a smooth invariant manifold of lower dimension than that of the full phase space [5]. Slightly beyond the critical point, at which the synchronized state becomes unstable for the perturbation out of the invariant manifold, the orbit escapes far away from the invariant manifold (burst or *on* state) but returns to its neighborhood and stays there for a long time (laminar phase or *off* state). This temporal evolution is repeated in an irregular manner. The name *on-off* intermittency comes from this behavior. On the other hand, most of the intermittency discussed so far, so-called Pomeau-Manneville intermittency (types I–III), is caused by linear instabilities of periodic trajectories [11,12].

Though the on-off intermittency is a deterministic phenomenon, the theory is founded on stochastic approaches [2,4]. In our previous paper [13] we started to construct a theory based on a deterministic approach, in which we proposed a map. It enabled us to obtain the exact expression of the distributions of burst amplitudes and of laminar durations, both of which characterize the on-off intermittency very well because of their power-law behaviors. We derive analytically the power spectrum of the solvable map in the present paper, which is expressed by a scaling function in the neighborhood of the critical point. We give then a plausible discussion and a numerical test that support that the scaling law is universal.

We can analytically obtain the power spectrum of some low-dimensional maps using the Frobenius-Perron (FP) operator. If the map is piecewise linear and has a Markov partition, the FP operator can be treated as a matrix [14]. The simplest case in which the on-off intermittency appears is a coupled system consisting of two chaotic oscillators. When the coupling is strong enough, the two oscillators are synchronized. As the coupling is decreased, the system reaches a critical point at which the synchronization is broken. Let X and x be the center-of-mass and relative coordinate, respectively. Just after the synchronization was broken, x behaves in an intermittent manner. This is called on-off intermittency. We proposed a model in which the dynamics of X and x is given by the following piecewise linear map:

$$X_{n+1}=\begin{cases} X_n/a, & 0\leqslant X_n<a,\\ \dfrac{1-X_n}{1-a}, & a\leqslant X_n<1,\end{cases}$$

$$x_{n+1}=\begin{cases} b^{-2}x_n, & 0\leqslant x_n<b^2,\ 0\leqslant X_n<a,\\ bx_n, & 0\leqslant x_n<b^2,\ a\leqslant X_n<1,\\ \dfrac{b-x_n}{b(1-b)}, & b^2\leqslant x_n<b,\\ \dfrac{x_n-b}{1-b}, & b\leqslant x_n<1,\end{cases}$$

where $b=[a/(1-a)]^{1/3}$. The detailed derivation of the model from a general coupled system is given by [13]. We expand the manifold in the vicinity of the invariant one causing the on-off intermittency, and take the lowest order. We also add the mechanism of reinjection, which enables us to treat the whole model analytically. The reinjection process occurs at a distance of the invariant manifold. Thus it does not affect the characteristic statistics of the on-off intermittency. We divide the phase space $R=\{(X,x)|0\leqslant X<1,0\leqslant x<1\}$ into the domains $R_{i,j}$ $(i=1,2;j=1,2,\ldots,\infty)$ defined as $R_{1,j}=\{(X,x)|0\leqslant X<a,b^j\leqslant x<b^{j-1}\}$, $R_{2,j}=\{(X,x)|a\leqslant X<1,b^j\leqslant x<b^{j-1}\}$. Set $R_j=R_{1,j}\cup R_{2,j}$, and we have $R=R_1\cup R_2\cup R_3\cdots$. The domain $R_{1,j}$ is mapped into R_{j-2} and R_{j+1}, $R_{2,j}$ $(j\geqslant 3)$ into R_{j+1}. The domains R_1 and R_2 are mapped into R. Thus the domains $R_{1,j}$ and $R_{2,j}$ construct a Markov partition of R. The elements of the FP matrix of this system are given as follows:

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$$H_{j,1}=1-b, \quad j\geqslant 1,$$

$$H_{j,2}=b(1-b), \quad j\geqslant 1,$$

$$H_{j,j+2}=ab^2, \quad j\geqslant 1,$$

$$H_{j,j-1}=(1-a)b^{-1}, \quad j\geqslant 4,$$

$$H_{i,j}=0, \quad \text{otherwise.}$$

The FP matrix always has a nondegenerate eigenvalue of unity [15]. Let the corresponding right and left eigenvector h_j and v_j, respectively. The probability density $P(x)$ is given by $P(x)=\Sigma_{j=1}^{\infty}h_jE_j(x)$, where the characteristic function $E_j(x)$ is equal to 1 for $x\in R_j$ and 0 otherwise. The element v_j corresponding to the width of R_j in the x direction is given by $v_j=b^{j-1}(1-b)$. The element h_j is given by $h_1=r(q+\nu_1)$, $h_2=r(q+\nu_2)$, and $h_j=r(q+\nu^{j-3})$ for $j\geqslant 3$, where $a_c=1/3$, $a=a_c(1+\epsilon)$, $b=[a/(1-a)]^{1/3}$, $q=-b(1-ab^2\nu^2)/(1-a)$, $\nu=[\sqrt{(4-3a^2)a}-a]/(2ab)$, $\nu_1=[1+ab^2(1-\nu^2)]$, $\nu_2=[1+ab^2\nu(1-\nu)]$, and $r=[q+(1-b)\nu_1+b(1-b)\nu_2+b^2(1-b)/(1-b\nu)]^{-1}$ By use of the FP matrix the power spectrum $I(\omega)$ is given by

$$I(\omega)=\sum_{l,m} v_l u_l(2\,\mathrm{Re}[R_{lm}]-\delta_{lm})u_m h_m,$$

where R_{lm} is the lm element of $1/(E-zH)$ [14]. The unit matrix and the FP matrix are denoted by E and H, respectively, and $z=\exp[i\omega]$. The variable u_l is the lth element of the observable under consideration. The real part of Z is denoted by $\mathrm{Re}[Z]$.

For simplicity, we consider hereafter the binary time series in which $u_l=1$ for $l=1,2$, $u_l=0$ for $l\geqslant 3$. The laminar and burst phase correspond to 0 and 1, respectively. In this case we need the four elements R_{11}, R_{12}, R_{21}, R_{22} only. The element R_{ij} is given by the adjoint of the ij element of $E-zH$ divided by $|E-zH|$. It can be shown that $|E-zH|$ is given by the relation $|E-zH|=(1+\alpha+\beta)A_{m-2}-\alpha\gamma B_{m-2}+\beta\gamma C_{m-2}$ for the m-dimensional truncated matrix, where $\alpha=-z(1-b)$, $\beta=b\alpha$, $\gamma=-zab^2$, $\delta=-z(1-a)/b$. In the same way, the adjoint of the (1,1) element is given by $(1+\beta)A_{m-2}+\beta\gamma C_{m-2}$, so that we have

$$R_{11}=[(1+\beta)A_{m-2}+\beta\gamma C_{m-2}]/[(1+\alpha+\beta)A_{m-2}-\alpha\gamma B_{m-2}+\beta\gamma C_{m-2}]. \tag{1}$$

Each of A_m, B_m, and C_m is the determinant of a definite m-dimensional matrix, and can be expressed by the determinants of the lower dimensional matrices, which is summarized as

$$A_{m+3}-A_{m+2}-\gamma\delta^2A_m=0,$$

$$B_{m+3}+\gamma B_{m+1}-\gamma^2\delta B_m=A_{m+2}+\gamma\delta A_m,$$

$$C_{m+3}+\gamma C_{m+1}-\gamma^2\delta C_m=(\delta-1)A_{m+1}-\gamma\delta A_m.$$

The solutions A_m of the third-order linear homogeneous difference equation are written in the form $A_m=A_a\rho_1^m+A_b\rho_2^m+A_c\rho_3^m$ ($|\rho_1|\geqslant|\rho_2|\geqslant|\rho_3|$), where ρ_1, ρ_2, and ρ_3 are the solutions of the characteristic equation $\rho^3-\rho^2-\gamma\delta^2=0$ and A_a, A_b, and A_c can be expressed by first three terms. We have $A_m\sim A_a\rho_1^m$ for $m\to\infty$. The solutions B_m and C_m of the third-order linear inhomogeneous difference equation are expressed by the general solution added by the special solution. Since each of the moduli of the solutions of the characteristic equation for B_m and C_m is shown to be less than $|\rho_1|$, only the contribution of the special solution is appreciable for $m\to\infty$. Following the method of variation of constants, we substitute $B_m=B\rho_1^m$ into the difference equation for B_m, and hereafter we denote ρ_1 by ρ, so that we have $B=B_fA_a$, $B_f=[\rho^2+\gamma\delta]/[\rho^3+\gamma\rho+\delta\gamma^2]$. In the same way for C_m we have $C=C_fA_a$, $C_f=[(\delta-1)\rho-\gamma\delta]/[\rho^3+\gamma\rho+\delta\gamma^2]$. When we take the contribution of ρ^m only into account for $m\to\infty$, Eq. (1) reduces to $R_{11}=[(1+\beta)+\beta\gamma C_f]/[(1+\alpha+\beta)-\alpha\gamma B_f+\beta\gamma C_f]$, where the common factor $A_a\rho^{m-2}$ was removed. In the same way we have $R_{12}=[-\beta+\beta\gamma B_f]/[(1+\alpha+\beta)-\alpha\gamma B_f+\beta\gamma C_f]$, $R_{21}=[-\alpha-\alpha\gamma C_f]/[(1+\alpha+\beta)-\alpha\gamma B_f+\beta\gamma C_f]$, $R_{22}=[1+\alpha-\alpha\gamma B_f]/[(1+\alpha+\beta)-\alpha\gamma B_f+\beta\gamma C_f]$. Thus we have the analytical expression of the power spectrum

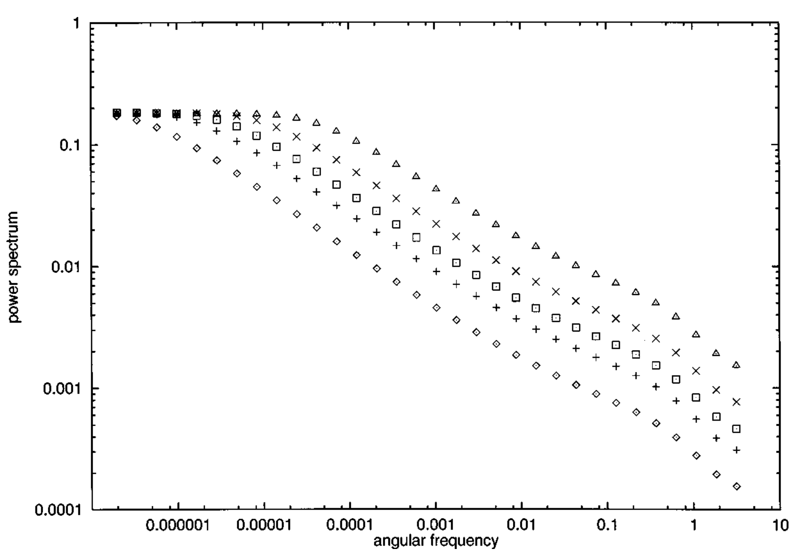

FIG. 1. Power spectra of the solvable map at $\epsilon=0.001$ ($\diamond$), 0.002 (+), 0.003 ($\square$), 0.005 ($\times$), 0.01 ($\triangle$).

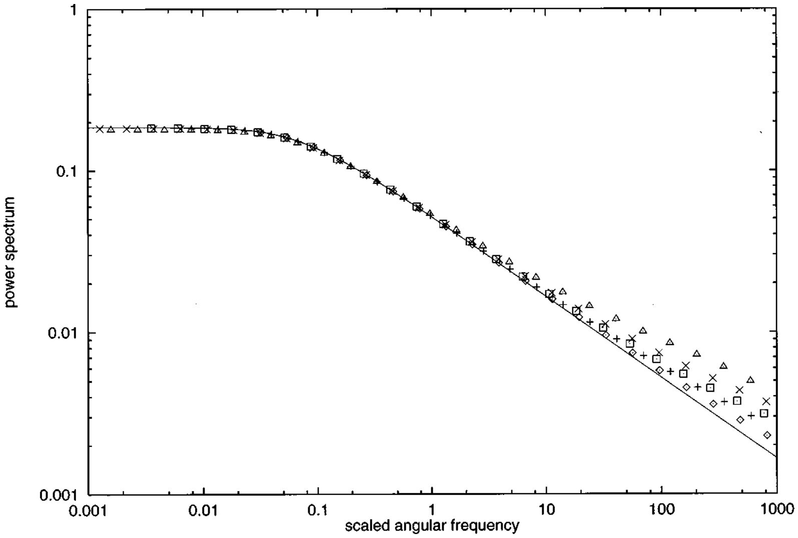

FIG. 2. Scaling function and the original power spectra of the solvable map whose angular frequency is scaled by $\omega_c = 2\pi\epsilon^2$ at $\epsilon = 0.001$ ($\diamond$), 0.002 (+), 0.003 ($\square$), 0.005 ($\times$), 0.01 ($\triangle$).

$$I(\omega) = (1-b)(h_1 - bh_2)(2\,\mathrm{Re}[I_{\mathrm{num}}/I_{\mathrm{den}}]-1),$$

$$I_{\mathrm{num}} = \rho^2 + \gamma\rho + \gamma\delta(\gamma+\delta),$$

$$I_{\mathrm{den}} = (1+\alpha+\beta-\alpha\gamma)\rho^2 + \gamma(1+\alpha+\beta\delta)\rho + \gamma\delta(\gamma+\delta+\delta\alpha+\delta\beta).$$

Figure 1 depicts the power spectra at $\epsilon = 0.001$, 0.002,0.003,0.005,0.01.

Taking the lowest order of ϵ and ω, we have

$$I(\omega) = \frac{f}{\sqrt{2}}\mathrm{Re}\left[\frac{\epsilon/2}{\sqrt{\epsilon^2/4 - i\omega} - \epsilon/2}\right],$$

$$f = 4(53t^2 - 42t - 30)/[9\sqrt{2}(-t^2+4t-2)] \approx 0.263046,$$

$$t = 2^{1/3},$$

which can be expressed by the scaling function

$$I(x) = \frac{f}{\sqrt{\sqrt{1+(8\pi x)^2}+1}}, \qquad (2)$$

where the scaled angular frequency $x = \omega/\omega_c$ were introduced. As we will show later, the characteristic time is given by $\tau_c = \sigma/\Lambda^2$, with the transverse Lyapunov exponent $\Lambda \sim (\ln 2/3)\epsilon$ and its variance $\sigma = (\ln 2/3)^2 - (\ln 2/3)\epsilon$ for $\epsilon \to 0$, so that we have $\tau_c = 1/\epsilon^2$ and $\omega_c = 2\pi/\tau_c = 2\pi\epsilon^2$. This scaling function and the original power spectra whose angular frequency is scaled by ω_c are shown in Fig. 2.

The scaling function behaves as $f/\sqrt{2}$ for $x \ll 1$ and $f/\sqrt{x}$ for $x \gg 1$. The latter is nothing but the –1/2 power-law decay of the power spectrum known as one of the remarkable characteristics of the on-off intermittency, which was also derived on the basis of a stochastic approach [2]. Here we found furthermore the existence of the characteristic time scale $1/\omega_c$ and the related scaling law, which implies that the self-similarity of the time series of the on-off intermittency holds only for the shorter time scale than $1/\omega_c$.

Is the existence of the characteristic time scale $1/\omega_c$ specific to the model or universal? Let us consider the average and the variance of the local expansion rates in the transverse direction to the subspace in which the synchronized solution is located, which is respectively denoted by Λ and σ in the following. The former is called transverse Lyapunov exponent. In the present model Λ is given by the average of $\ln|dx_{n+1}/dx_n|$. There are two length scales Λt and $\sigma t^{1/2}$. The former measures the linear growth of the burst, while the latter the Brownian-motion-like fluctuation. Equating these two length scales, we obtain the characteristic time scale $t = \tau_c = \sigma/\Lambda^2$ by which the Brownian motion and the linear growth are roughly separated. Above discussions are not dependent on details of the model concerned. Thus we believe that the existence of the characteristic time scale τ_c is universal.

To confirm this universality we perform the following numerical test. We use the stochastic model which that used by Fujisaka and Yamada [16] and simple coupled maps. The stochastic model is given by $x_{n+1} = x_n \exp(\epsilon - x_n + \zeta_n)$, where ϵ is a control parameter and ζ_n is a random variable with $\langle\zeta_n\rangle_T = 0$, $\langle\zeta_n\zeta_{n'}\rangle_T = \delta_{nn'}$, where $\langle\cdots\rangle_T$ means the long time average. For $0 < \epsilon \ll 1$, x_n shows on-off intermittency.

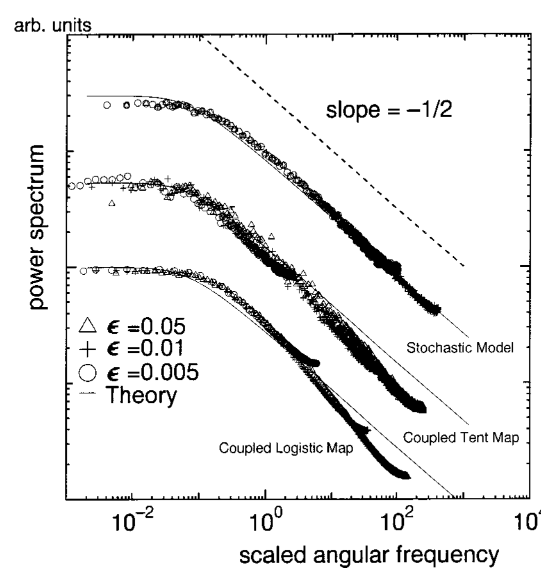

FIG. 3. Scaling function and the scaled power spectra of the numerical models at $\epsilon = 0.005,0.01,0.05$.

The coupled maps are $y_{n+1}=(1-c)g(y_n)+cg(z_n)$, $z_{n+1}=(1-c)g(z_n)+cg(y_n)$, where c is a coupling intensity and $g(y)$ is the mapping function given either by $g(y)=y/a$, $(0\leqslant y<a)$, $(1-y)/(1-a)$, $(a\leqslant y<1)$ with $a=0.2$ (tent map) or by $g(y)=r-y^2$ with $r=1.7$ (logistic map). When the coupling c is greater than the critical value c_*, the synchronized motion is stable, and for $\epsilon\equiv(c_*-c)/c_*\ll 1$ the on-off intermittency is observed in the relative coordinate $x_n\equiv y_n-z_n$. For three models, the power spectra of x_n are numerically calculated from $I(\omega)=\langle|\Sigma_{n=0}^{T-1}x_n e^{-i\omega n}|^2/T\rangle$, where $T=2^{22}$ and $\langle\cdots\rangle$ denotes the initial ensemble average over 1000 points. In Fig. 3, we show the theoretical scaling function and numerical power spectra at $\epsilon=0.05,0.01,0.005$.

This numerical result supports the universality of the scaling law of the power spectrum with the characteristic time scale τ_c found in the solvable map. The coincidence between the theoretical and numerical scaling functions is relatively good for the coupled tent map and for the stochastic model. A remarkable deviation is observed for the logistic map. In this case the power spectrum does not follow even the –1/2 power-law decay that is a well-known feature of the on-off intermittency [2–5]. However, the scaling law itself holds clearly. It should be made clear in the future where the deviation comes from. Nonhyperbolicity or long time correlation is thought to be an important factor.

In fact, Fujisaka and Yamada analytically derived the power spectrum of the on-off intermittency for a multiplicative noise system, which coincides with Eq. (2) [17]. Thus it has been shown analytically as well as numerically that the scaling law holds in both deterministic and stochastic systems. We may conclude that the scaling law with the characteristic time scale τ_c is universal. (The universality of the –1/2 power-law decay has been recognized on the stochastic approaches and the discussions of temporal fluctuations in the transverse Lyapunov exponent [2,4,5].) We hope that this scaling law will be confirmed experimentally. We thank Hirokazu Fujisaka, Takehiko Horita, Masayoshi Inoue, Arkady Pikovsky, and Stefan Thomae for illuminating discussions.

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